1
Fundamental concepts

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Exercise 1.1: Uniform distributions

$$
\begin{aligned}
P(x) &= \frac{x}{w}\,; \\
E[x] &= \frac{1}{w}\int_{-w/2}^{w/2} x\,dx = 0\,; \\
E[x^2] &= \frac{1}{w}\int_{-w/2}^{w/2} x^2\,dx = \frac{1}{w}\left[\frac{x^3}{3}\right]_{-w/2}^{w/2} = 2\frac{w^2}{24}\,; \\
\sigma &= \sqrt{E[x^2]-E[x]^2} = \frac{w}{\sqrt{12}}\,.
\end{aligned}
$$

Exercise 1.2: Poisson distributions 1

$$
\begin{aligned}
\phi(u) &= \sum_0^\infty \frac{e^{iru}e^{-\lambda}\lambda^r}{r!} \\
&= e^{-\lambda}\sum_0^\infty \frac{(\lambda e^{iu})^r}{r!} = e^{-\lambda}e^{\lambda e^{iu}} = e^{\lambda e^{iu}-\lambda} = e^{\lambda(e^{iu}-1)}\,.
\end{aligned}
$$

Exercise 1.3: Poisson distributions 2

A solution “by hand” looks like this:

$$
e^{-3.7} + 3.7e^{-3.7} + \frac{3.7^2}{2}e^{-3.7} = 0.2854331\,.
$$

In R, do the following:

```
> ppois(2,3.7)
[1] 0.2854331
```

In ROOT, you can use

```
root[1] ROOT::Math::poisson_cdf(2,3.7)
<double> 2.85433113100068300e-001
```

Exercise 1.4: Bayes' theorem

$$P(\text{null}|\text{X exists}) = \frac{N-1}{N} \qquad P(\text{null}|\text{X does not exist}) = 1\,,$$
$$P' = \frac{\frac{N-1}{N}}{P\frac{N-1}{N} + (1-P)} P = \frac{N-1}{N-P} P\,.$$

If $N = 10$, P=0.9 and the first result is blank, then $P' = \frac{9}{9.1} \times 0.9 = 0.891$. The second failure gives $P'' = \frac{8}{8.109} \times 0.891 = 0.878$. Repeating for nine failures gives 0.473. The same sequence with P=0.99 starts 0.9889, 0.9875, and finishes 0.9082.

Conclusion: With one last chance after nine failures, someone who is fairly certain (90%) of a proposition has their belief reduced below 50%. However, someone who is very strongly (99%) convinced still believes at the 90% level.

Exercise 1.5: p-values

For 1-tailed, use `qnorm(.9)`, `qnorm(.95)`, `qnorm(.99)` to get 1.281, 1.644 and 2.326.

For 2-tailed, use `qnorm(.95)`, `qnorm(.975)`, `qnorm(.995)` to get 0.842, 1.281 and 2.054.

Exercise 1.6: Jeffreys prior

Start with

$$\ln L = \ln e^{-\nu} \frac{\nu^r}{r!} = -\nu + r \ln \nu - \ln(r!)\,.$$

Differentiating once gives

$$\frac{d \ln L}{d\nu} = -1 + \frac{r}{\nu}\,,$$

and twice:

$$\frac{d^2 \ln L}{d\nu^2} = -\frac{r}{\nu^2}\,.$$

Then negate and take the expectation value:

$$-E[\frac{d^2 \ln L}{d\nu^2}] = \frac{1}{\nu^2} E[r] = \frac{1}{\nu}\,,$$

and the square root gives

$$\frac{1}{\sqrt{\nu}}\,.$$