8
How to deal with systematic uncertainties

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A hypothetical B-meson decay

a) The sum of events in the signal sidebands is

$$N_{\text{sideband}} = N_{\text{sideband}}^{4.68-4.98\ \text{GeV}} + N_{\text{sideband}}^{5.58-5.98\ \text{GeV}} = 1417 + 2846 = 4263 \qquad (8.1)$$

With this and assuming a linear behaviour of the background, the expected number of background events in the signal region from 5.13 GeV to 5.33 GeV is

$$N_{\text{bkg}} = N_{\text{sideband}} \cdot \frac{\text{width of signal region}}{\text{width of sideband regions}} = 4263 \cdot \frac{0.3\ \text{GeV}}{2 \cdot 0.3\ \text{GeV}} = 2131.5 \pm 32.6, \quad (8.2)$$

where the error on the estimate comes from the sideband statistics (it is $\frac{1}{2}\sqrt{N_{\text{bkg}}}$). The background estimate in the signal region as extrapolated from the sidebands is shown in figure 8.1. Judging by eye, it seems that the background is slightly underestimated — possibly indicating a non-linear behaviour, which should be taken into account by a systematic uncertainty (see below)[1].

The only possibility of a systematic uncertainty on the background estimate is a possible non-linear behaviour of the background. Two possibilities should always be considered:

(1) A smooth, but possibly non-linear shape of the background. In this case a higher order polynomial or other smooth functions could be assumed to describe the background. Such functions can be fitted in the sideband regions and then extrapolated into the signal region. This would change the factor, with which N_{sideband} is multiplied in equation 8.2, thus leading to a slightly different result. The difference to the original result could then be quoted as systematic uncertainty arising from the background shape.

(2) A "peaking background" which contributes only to the signal region. Such a background cannot be assessed by sideband subtraction. One therefore needs to find other means as, for example, Monte Carlo simulations to investigate any possible background peaking underneath the signal.

1) In fact the eye is being fooled in this example, as an almost linear background had been generated. Nevertheless, in a real experiment the background shape is usually not a-priori known, and then a different shape always needs to be considered.

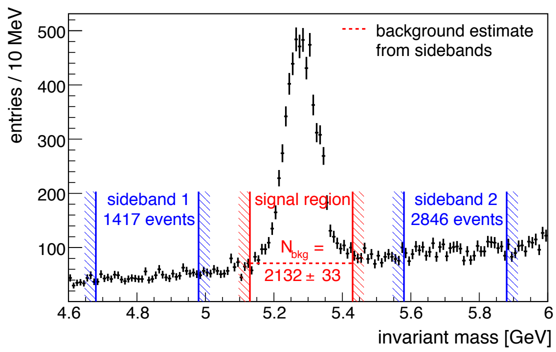

Figure 8.1 Background estimation by sideband subtraction (exercise 8.1a)).

b) If you trust the Monte Carlo simulation on the mass resolution, the best choice for the mass cut is given by the smallest total uncertainty. This is the case for the smallest value of $\sqrt{N_{\text{sig}} + N_{\text{bkg}}}/N_{\text{sig}}$ with the expected numbers N_{sig} and N_{bkg} of signal and background events in the signal region. Since N_{sig} is not known (it is the outcome of the measurement!), the figure of merit to be optimised is $\sqrt{N_{\text{bkg}}}/\mathcal{A}$ with the signal acceptance $\mathcal{A}$. Evaluating N_{bkg} from sideband subtraction and computing $\mathcal{A}$ by assuming a Gaussian signal around $m_B = 5.28$ GeV with $\sigma_{\text{sig}} = 0.05$ GeV yields for mass cuts between $\pm\,0.01$ and $\pm\,0.20$ GeV around m_B the values in table 8.1 and figure 8.3, respectively. The best measurement would thus be made, when applying a mass cut of $\pm\,0.06$ GeV around the nominal B mass, corresponding to just 1.2 standard deviations on the invariant mass distribution (see figure 8.2). The signal acceptance would just be 77 % and the resolution on the invariant mass needs to be precisely simulated to yield the correct acceptance. A mass cut as tight as this is therefore usually not recommended.
If you do not trust the simulated mass resolution a mass cut of $\pm\,0.15$ GeV, corresponding to $\pm\,3\,\sigma$, is a usual choice. This cut has an inefficiency of only 0.3 % (for a perfect Gaussian), therefore small uncertainties in the mass resolution would not affect the final result.

c) A different mass resolution changes the signal acceptance when the mass cut is unchanged. Using a given signal Monte Carlo sample, the effect of a faulty resolution can be assessed as it is explained in chapter 8.4.2.2: For each Monte Carlo event with reconstructed mass m additional histograms are filled with $m \rightarrow m + k \cdot (m_B - m)$, where k is the relative change in the mass resolution.
Let us assume that the invariant mass follows a Gaussian resolution with width $\sigma = 0.05$ GeV, which was determined from a sample of Monte Carlo events. Applying the above procedure for resolutions of $\pm\,10$ % therefore results in new mass distributions of these MC events with $\sigma = 0.055$ GeV and 0.045 GeV, respect-

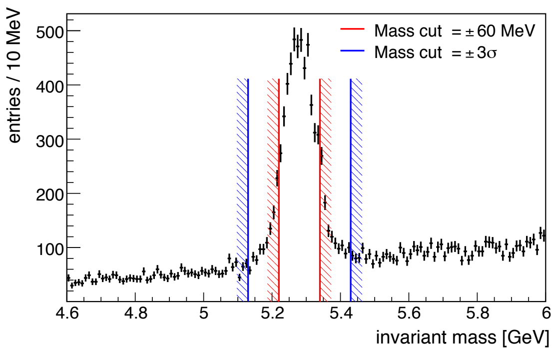

Figure 8.2 Different cuts on the invariant mass (exercise 8.1b)).

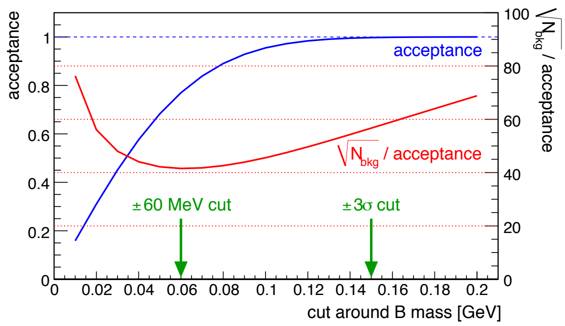

Figure 8.3 Signal acceptance and figure of merit $\sqrt{N_{\rm bkg}}/\mathcal{A}$ for different applied cuts on the invariant mass (exercise 8.1b), values from table 8.1).

ively[2]. A comparison between the three different simulated resolutions is given in figure 8.4: Although the nominal resolution seems to fit the data best, the variations still look reasonable; a variation of ± 10 % therefore is a good choice for estimation of systematics.

The resulting systematic uncertainties together with the statistical uncertainties, acceptances, and branching fractions for both mass cuts of $2\,\sigma^{\rm MC}$ and $3\,\sigma^{\rm MC}$ are given in table 8.2.

It is clearly seen, that despite the larger statistical error, the $\pm 3\,\sigma$ cut provides a much smaller total uncertainty. In fact, one would have to know the resolution to better than ± 2 % to profit from the slightly better statistical error of a $\pm 2\,\sigma$ mass cut.

2) Note, that this is different then just simulating new events with a different resolution: In our case the three distributions (± 10 % and the original) are completely correlated, since the same original events were used! Therefore no statistical uncertainties occur in the study.

Mass cut [GeV]	$N_{\text{bkg}} \pm \sigma_{N_{\text{bkg}}}$	Acceptance $\mathcal{A}$	$\sqrt{N_{\text{bkg}}}/\mathcal{A}$
± 0.01	142.1 ± 2.2	0.1588	76.3
± 0.02	284.2 ± 4.4	0.3103	56.1
± 0.03	426.3 ± 6.5	0.4510	48.0
± 0.04	568.4 ± 8.7	0.5758	44.1
± 0.05	710.5 ± 10.9	0.6825	42.2
± 0.06	852.6 ± 13.1	0.7697	41.6
± 0.07	994.7 ± 15.2	0.8383	41.8
± 0.08	1136.8 ± 17.4	0.8904	42.6
± 0.09	1278.9 ± 19.6	0.9283	43.9
± 0.10	1421.0 ± 21.8	0.9549	45.6
± 0.11	1563.1 ± 23.9	0.9723	47.5
± 0.12	1705.2 ± 26.1	0.9838	49.7
± 0.13	1847.3 ± 28.3	0.9908	51.9
± 0.14	1989.4 ± 30.5	0.9949	54.3
± 0.15	2131.5 ± 32.6	0.9973	56.7
± 0.16	2273.6 ± 34.8	0.9986	59.1
± 0.17	2415.7 ± 37.0	0.9993	61.6
± 0.18	2557.8 ± 39.2	0.9997	64.0
± 0.19	2699.9 ± 41.3	0.9998	66.4

Table 8.1 Variation of the background estimation N_{bkg}, signal acceptance $\mathcal{A}$, and the figure of merit $\sqrt{N_{\text{bkg}}}/\mathcal{A}$ with different applied cuts on the invariant mass (exercise 8.1b)). The N_{bkg} values were estimated by sideband subtraction (see previous exercise 8.1a)).

d) The fit function is

$$f(m) \;=\; N_{\text{sig}} \cdot BW \cdot \frac{1}{\sqrt{2\pi\,\sigma^2}} \cdot e^{-\dfrac{(m-m_B)^2}{2\,\sigma^2}} \;+\; a \;+\; b \cdot (m - m_B) \qquad (8.3)$$

with the free parameters $N_{\text{sig}} = N - N_{\text{bkg}}$ = number of background subtracted signal events and offset a and slope b of the linear function for the background. The bin width BW is 0.01 GeV and the multiplication automatically takes care of the correct normalization.

With a least-squares fit of a single Gaussian with fixed mean $m_B = 5.28$ GeV and width $\sigma = 50$ MeV (shown in figure 8.5) the result for the number of signal events is $N_{\text{sig}} = 4951.0 \pm 82.8$, directly corresponding to a branching ratio of $\mathcal{B} = (4.95 \pm 0.08) \cdot 10^{-6}$. This value is very consistent with the result obtained in the previous exercise with a sideband subtraction and slightly more precise. The somewhat higher accuracy is of course not surprising, since we have added the information on the signal shape to the analysis. A systematic uncertainty on the background shape can for instance be estimated by adding a quadratic term $+\,c \cdot (m - m_B)^2$ to

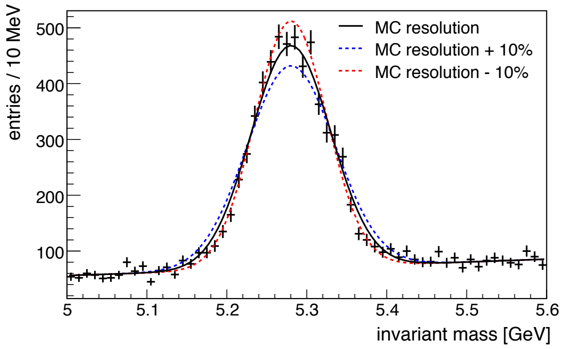

Figure 8.4 Comparison of different simulated mass resolutions (exercise 8.1c)).

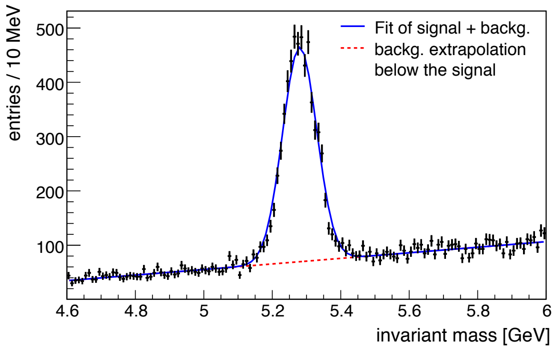

Figure 8.5 Fit of signal and background with a Gaussian plus a linear background. (exercise 8.1d)).

the background in equation (8.3) which results in $N_{\text{sig}} = 4978.5 \pm 85.9$ fitted signal events. By using the difference to the original result we can estimate a systematic error of 27.5 events, corresponding to $\sigma^{\mathcal{B}}_{\text{bkg syst}} = \pm 0.03 \cdot 10^{-6}$.
By performing a fit to the signal we may introduce an additional source of systematics, which is a possible uncertainty on the signal position and shape. In general both the absolute mass scale and the mass resolution of the real data might not be perfectly simulated. Therefore often not only the signal size but also its mean and width are left free in the fit to become independent of the simulation. Releasing m_B and the resolution σ in the fit results in $N_{\text{sig}} = 4862.9 \pm 84.5$ fitted signal events (assuming a linear background shape) and a fitted width of $\sigma = 46.4 \pm 0.8$ MeV. Both values are significantly different from the previously fitted signal estimate and the assumed width of 50 MeV, pointing to problem in the MC description of the signal width. One example of such a problem would be tails of the signal distribution (sometimes modeled by a second Gaussian with a width $\gg \sigma$), which would not be

	Mass cut of $\pm 2\,\sigma = \pm 0.10$ GeV	Mass cut of $\pm 3\,\sigma = \pm 0.15$ GeV
Acceptance (-10 % resolution)	0.974	0.999
Acceptance (nominal)	0.955	0.997
Acceptance ($+10$ % resolution)	0.931	0.994
Relative acceptance systematics	± 0.0224	± 0.0027
Relative background systematics	± 0.0046	± 0.0067
Relative statistical uncertainty	± 0.0166	± 0.0171
Total relative uncertainty	± 0.0046	± 0.0067
Number N_{sig} of signal events	4702 ± 78	4903 ± 84
Branching fraction $\mathcal{B}$ $[10^{-6}]$	$4.92 \pm 0.08_{\text{stat}}$ $\pm 0.02_{\text{bkg}} \pm 0.11_{\text{acc}}$ $= \mathbf{4.92 \pm 0.14}$	$4.92 \pm 0.08_{\text{stat}}$ $\pm 0.03_{\text{bkg}} \pm 0.01_{\text{acc}}$ $= \mathbf{4.92 \pm 0.09}$

Table 8.2 Acceptances for nominal and by 10 % increased/decreased mass resolutions and the corresponding systematic uncertainties for mass cuts of $\pm 2\,\sigma$ and $\pm 3\,\sigma$ around m_B. Given are also the statistical, background, and total uncertainties as well as the resulting branching fractions. The systematics on the sideband estimated background (see exercise 8.1c)) includes only the statistical uncertainty — it does not include possible uncertainties on the shape. The branching fractions were computed using the formula given in the exercise.

taken into account by a fit with a single Gauss function. Here one could introduce a second Gaussian to the fit and quote the difference as systematic uncertainty.

Additional remark: Revealing of the simulated parameters At the very end, we should resolve how the hypothetical signal and background events were generated.

- In total 5000 signal events were generated according to a double Gaussian signal distribution

$$f_{\text{signal}}(m) = 0.9 \cdot \frac{1}{\sqrt{2\pi\,\sigma_1^2}} \cdot e^{-\frac{(m-m_B)^2}{2\,\sigma_1^2}} + 0.1 \cdot \frac{1}{\sqrt{2\pi\,\sigma_2^2}} \cdot e^{-\frac{(m-m_B)^2}{2\,\sigma_2^2}} \qquad (8.4)$$

  with the widths $\sigma_1 = 45$ MeV and $\sigma_2 = 80$ MeV of the double Gaussian (see figure 8.6). If this double Gaussian structure is not known before (e.g. from a good detector simulation), it is prone to give a bias on the acceptance, if the mass cut is chosen too tight: For the "optimal" cut of ± 60 MeV in exercise 8.1b) the true acceptance would be underestimated by 2.7 %, while a $3\,\sigma$ mass cut would only result in an error of 0.4 % when assuming a mass resolution of 50 MeV.
  With the assumed flux of 10^{10} B decays and an acceptance of 10 % (not including

any mass cut) these 5000 signal events correspond to a branching fraction of $\mathcal{B} = 5 \cdot 10^{-6}$.

- The background events were generated according to a 2. order polynomial:

$$f_{\text{bkg}}(m) = 1 + 0.7 \cdot (m - m_B) + 0.1 \cdot (m - m_B)^2 \tag{8.5}$$

In total 10 000 events were generated, of which 2067 fall into the signal region of $\pm 3\,\sigma = \pm 150$ MeV around the B mass. Apparently the quadratic term is so small in our example, that it does not cause any significant difference between a linear or quadratic background description (see figure 8.6).

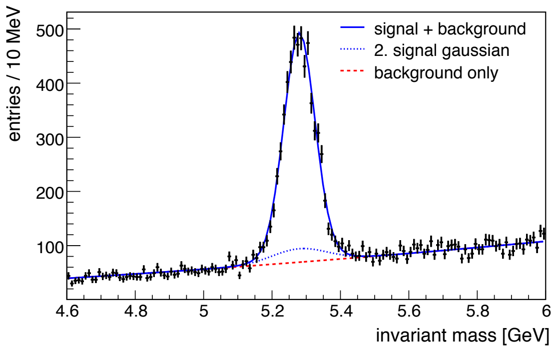

Figure 8.6 Data distribution with the true signal and background distributions used for the generation (equations (8.4) and (8.5)).