10
Statistical methods commonly used in high energy physics

Carsten Hensel and Kevin Kröninger

Exercise 10.1: Properties of an estimator

The mean value μ and variance V of $N \geq 2$ samples drawn from a uniform distribution in the interval $[0, 1]$ are

$$
\begin{aligned}
\mu &= 0.5\,, \\
V &= 1/12 \approx 0.0833\,.
\end{aligned}
$$

The mean value and variance can be estimated by the sample mean $\bar{x}$ and sample variance s'^2 (without Bessel correction) or s^2 (with Bessel correction),

$$
\begin{aligned}
\hat{\mu} &= \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x\,, \\
\hat{V} &= s'^2 = \frac{1}{N}\sum_{i=1}^{N} (x - \bar{x})^2\ , \\
\hat{V} &= s^2 = \frac{1}{N-1}\sum_{i=1}^{N} (x - \bar{x})^2\ .
\end{aligned}
$$

Using 100 000 sets of random numbers and $N = 10$ samples, the expected estimators (for the particular random seed chosen in the code) are

$$
\begin{aligned}
E[\bar{x}] &= 0.4998 \pm 0.0003\,, \\
E[s'^2] &= 0.0750 \pm 0.0001\,, \\
E[s^2] &= 0.0833 \pm 0.0001\,.
\end{aligned}
$$

As expected, the expected sample variances with and without Bessel correction differ significantly. The estimator s'^2 has a bias, the estimator s^2 does not.

Repeating the study with $N = 100$ samples yields

$$
\begin{aligned}
E[\bar{x}] &= 0.4999 \quad \pm 0.0001\,, \\
E[s'^2] &= 0.08252 \pm 0.00002\,, \\
E[s^2] &= 0.08335 \pm 0.00002\,.
\end{aligned}
$$

The bias of the estimator s'^2 became smaller.

Exercise 10.2: Neutrinoless double-β decay

a) Assuming that the data are described by a Poisson distribution, the expected number of background events as well as the results of the analysis for three different run times are summarised in table 10.1. The estimators for each ensemble are calculated from the marginal posterior probability of a simple Poisson model with signal and background contributions, ν_s and ν_b, respectively,

$$p(\nu_s|N) \propto \int \mathrm{d}\nu_b \, p(N|\nu_s, \nu_b) p(\nu_s) p(\nu_b) \,, \tag{10.1}$$

where

$$p(N|\nu_s, \nu_b) \quad = \quad \frac{(\nu_s + \nu_b)^N}{N!} e^{-(\nu_s+\nu_b)} \,, \tag{10.2}$$

and the priors $p(\nu_s)$ ($p(\nu_b)$) is flat (Gaussian).

Table 10.1 Most likely and expected 95% limit, and the standard deviation for run times of one month, one year and five years. Note that the mode is estimated from a coarse histogram with a bin width of 5.

Time [months]	1	12	60
Background expectation	1.5	18	90
Most likely limit	2.5	12.5	17.5
Avg. 95% limit on signal	4.4	10.6	21.4
Std. of the 95%limit	1.2	3.6	7.0

b) For a run time of four years, one expects 72 background events. From 100 observed events, the 95% upper limit on the number of expected signal events is 46.2. The posterior probability for that case is shown in figure 10.1 (left). Repeating the analysis on pseudo-data generated under the assumption that only background events contribute, the average 95% upper limit and standard deviation are 19.2 ± 6.3. The distribution corresponding distribution is shown in figure 10.1 (right). The resulting p-value, i.e. the integral from 46.2 to infinity, is consistent with zero given the finite number of pseudo experiments.

c) Repeating the analysis with the assumption that the background level is known to 10%, the observed limit reduces to 49.7. The marginal probability of the signal contribution is shown in figure 10.2 (left). Since the strongest information on the rate comes from the observed number of events, the number of expected signal and background events is anti-correlated with a linear correlation coefficient of $\rho = -0.57$. The two-dimensional posterior probability is shown in figure 10.2 (right).

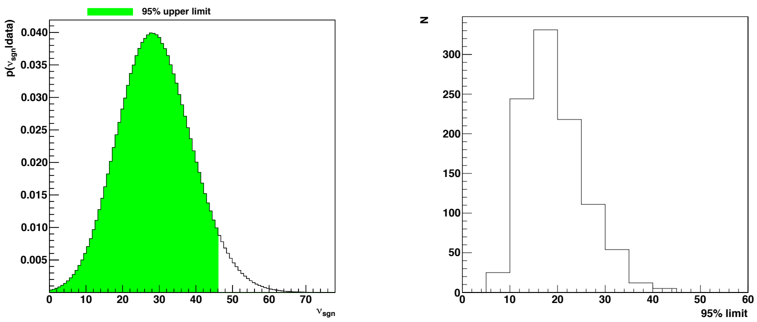

Figure 10.1 Left: Posterior probability for a Poisson model with an expectation of 72 background events and 100 observed events. Right: Distribution of the 95% limit obtained from pseudo experiments.

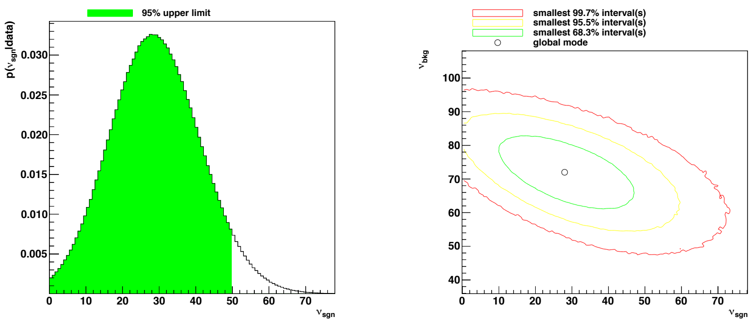

Figure 10.2 Left: Posterior probability for a Poisson model with an expectation of 72 ± 7.2 background events and 100 observed events. Right: Two-dimensional posterior probability density.

Exercise 10.3: Estimating a signal strength

It is assumed that the bin content of each bin fluctuates according to a Poisson distribution. The number of expected background events is 72 ± 7.2 in four years of run time. Analysing the observed spectrum with the three different likelihoods yields the following 95% upper limits on the number of expected signal events:

1) a) $\nu \leq 11.1$;
2) b) $\nu \leq 14.4$;
3) c) $\nu \leq \ \ 6.9$.